INVOLUTIONS OF KNOTS THAT FIX UNKNOTTING TUNNELS

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Abstract. Let K be a knot that has an unknotting tunnel τ. We prove that K admits a strong involution that fixes τ pointwise if and only if K is a two-bridge knot and τ its upper or lower tunnel.

1. Introduction

Let L be a link of one or two components in S^3. An *unknotting tunnel* for L is a properly embedded arc τ, with $L \cap \tau = \partial\tau$, such that the complement of a regular neighborhood of $L \cup \tau$ is a genus 2 handlebody. As described in [2], an unknotting tunnel induces a *strong inversion* of the link complement – that is, an involution of S^3 that sends each component of L to itself with reversed orientation.

When L is a two-component link, it is known that this involution can be chosen to fix τ pointwise. As a result, an argument of Adams in [1] gives us some geometric information about the tunnel: when the complement of L is hyperbolic, τ is isotopic to a geodesic in the geometric structure. [1] conjectures that the same is true for knots: that any unknotting tunnel for a hyperbolic knot is isotopic to a geodesic. As for links, this conjecture would follow easily if it were known that τ is fixed pointwise by some strong inversion of the knot.

The main result of this paper is that this happens only in the well-known special case of 2-bridge knots. (See Section 3 for background on on 2-bridge knots and their unknotting tunnels.)

Theorem 1.1. *Let $K \subset S^3$ be a knot with an unknotting tunnel τ. Then τ is fixed pointwise by a strong inversion of K if and only if K is a two-bridge knot and τ is its upper or lower tunnel.*

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devoted a great deal of time to hearing and vetting many versions of the proof-in-progress, and deserve my sincerest gratitude.

2. Tunnels and Involutions

Notation. From now on, K will denote a knot in S^3 that has an unknotting tunnel τ. Thus $K \cup \tau$ is realized as a graph with two vertices and three edges τ, K_1, and K_2, where $K = K_1 \cup K_2$. When identifying τ as a particular tunnel τ_0 of the knot K, we mean that $K \cup \tau$ is equivalent to $K \cup \tau_0$ via an isotopy of S^3 that preserves K setwise. Note that this notion of equivalence is stronger than isotopy of tunnels in the knot exterior, because the endpoints of τ are not allowed to pass through each other.

Definition. For any graph $\Gamma \subset S^3$, let $N(\Gamma)$ be an open regular neighborhood of Γ. We call $E(\Gamma) = S^3 \smallsetminus N(\Gamma)$ the *exterior* of Γ.

An unknotting tunnel induces a genus 2 Heegaard splitting of S^3 into handlebodies $V_1 = \overline{N(K \cup \tau)}$ and $V_2 = E(K \cup \tau)$. Let Σ be the Heegaard surface: $\Sigma = \partial N(K \cup \tau)$. A genus 2 handlebody admits a *hyper-elliptic involution* that preserves the isotopy class of every simple closed curve on its boundary; it is unique up to isotopy. Thus the hyper-elliptic involutions of V_1 and V_2 can be joined over Σ to an orientation-preserving involution φ_τ on all of S^3. This involution sends K to itself with reversed orientation; i.e. is a *strong inversion* of K (see [2]).

The action of φ_τ preserves the meridians of K_1 and K_2 on Σ while reversing the orientation on K – so it must switch the two vertices of $K \cup \tau$ and thus reverse the orientation on τ. It is conceivable, however, that some other involution ψ might fix τ pointwise while switching K_1 with K_2. Theorem 1.1 says that this only happens for 2-bridge knots.

3. Two-Bridge Knots

Definition. A *rational tangle* is a pair (B, t), where B is a 3-ball and t consists of two disjoint, properly embedded arcs γ_1 and γ_2. We further require that the γ_i are both isotopic to ∂B via disjoint disks D_i.

Definition. A knot $K \subset S^3$ is called a *two-bridge knot* if some sphere S splits (S^3, K) into two rational tangles. S is then called a *bridge sphere* for K.

A 2-bridge knot has a *4-plat projection* with two maxima at the top and two minima at the bottom [3]. Any horizontal plane, together with the point at infinity, then serves as a bridge sphere for K. The 4-plat projection also reveals two unknotting tunnels for K: an *upper tunnel*

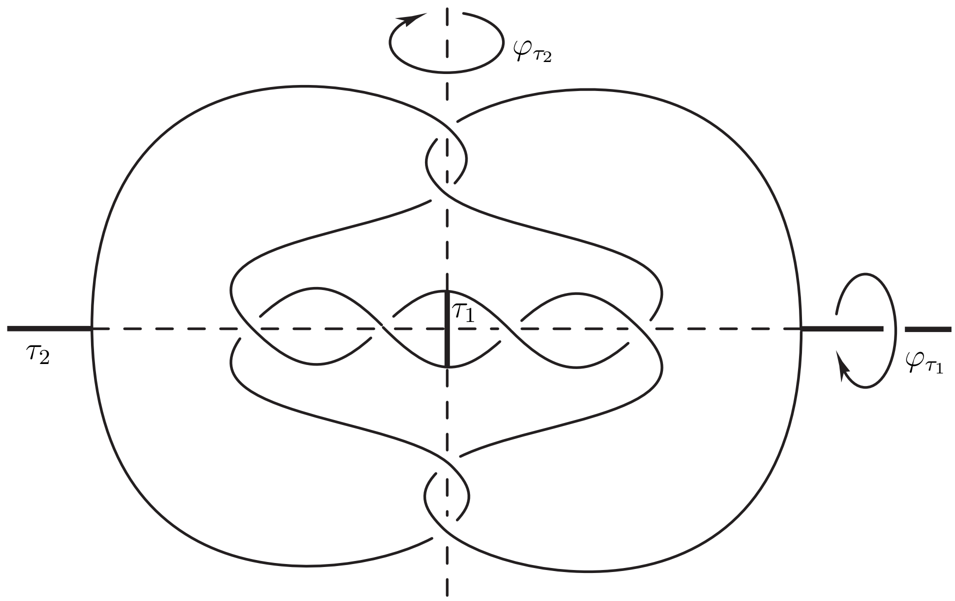

FIGURE 1. Upper and lower tunnels, and the corresponding involutions, in the tri-symmetric projection.

τ_1 connecting the two maxima, and a *lower tunnel* τ_2 connecting the two minima.

Two-bridge knots also have four *dual tunnels*, which may be isotopic to the upper or lower tunnels in special cases. Morimoto and Sakuma classified the six total tunnels up to homeomorphism and isotopy of the knot exterior [7]. Kobayashi showed that any unknotting tunnel of a 2-bridge knot is exterior-isotopic to one of these six [4].

The following result is an immediate consequence of [2, Section 3].

Lemma 3.1. *Let K be a two-bridge knot, and τ_1 and τ_2 be its upper and lower tunnels. Let φ_{τ_i} the involution of S^3 that comes from the handlebody decomposition induced by τ_i. Then φ_{τ_1} can be chosen to fix τ_2 pointwise, and φ_{τ_2} can be chosen to fix τ_1 pointwise.*

Proof. As described in [3], there is an isotopy of K from the 4-plat projection to a *tri-symmetric projection*, carrying τ_1 to part of the vertical axis of symmetry and τ_2 to part of the horizontal axis. (See Figure 1.) Abusing notation slightly, we continue to call these isotoped tunnels τ_1 and τ_2. Now, the involution φ_{τ_1} is evident in the figure as a 180° rotation about the horizontal axis containing τ_2, and thus fixes τ_2 pointwise. Similarly, φ_{τ_2} is a rotation about the vertical axis containing τ_1, fixing τ_1 pointwise. $\square$

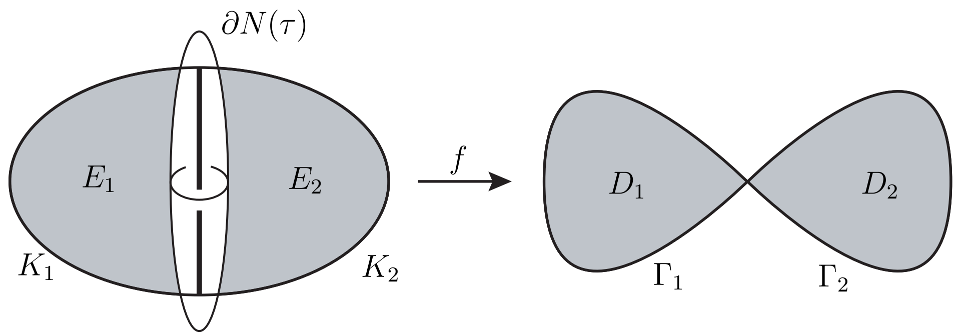

FIGURE 2. The graphs and disks of Corollary 3.3.

For the purpose of recognizing two-bridge knots based on involutions, our main tool is a theorem of Scharlemann and Thompson about planar graphs.

Definition. A finite graph $\Gamma \subset S^3$ is called *planar* if it lies on an embedded 2-sphere in S^3. Γ is called *abstractly planar* if it is homeomorphic to a planar graph.

Theorem 3.2. [8] *A finite graph $\Gamma \subset S^3$ is planar if and only if*
(1) Γ *is abstractly planar,*
(2) *every proper subgraph of* Γ *is planar, and*
(3) $E(\Gamma)$ *is a handlebody.*

Remark. For our purposes, we will only need the special case of this theorem when Γ consists of one vertex and two edges. This was first proved in an unpublished preprint of Hempel and Roeling.

Corollary 3.3. *Let K be a knot with unknotting tunnel τ, where τ splits K into edges K_1 and K_2. If both $K_1 \cup \tau$ and $K_2 \cup \tau$ are unknots, then K is a two-bridge knot with splitting sphere $\partial N(\tau)$. Furthermore, τ is an upper or lower tunnel.*

Proof. Let $f : S^3 \to S^3$ be a map that contracts $\overline{N(\tau)}$ to a point and is the identity outside a small regular neighborhood of $\overline{N(\tau)}$. Then $\Gamma = f(K \cup \tau)$ is a graph with one vertex and two edges, which is clearly abstractly planar. Each proper subgraph $\Gamma_i \subset \Gamma$ is the image of $K_i \cup \tau$, and is thus an unknotted circle. Also, $E(\Gamma)$ is a handlebody since $E(K \cup \tau)$ is a handlebody. Thus, by Theorem 3.2, Γ is planar.

Since Γ is planar, its subgraphs Γ_1 and Γ_2 bound disjoint disks D_1 and D_2 in S^3. (See Figure 2.) These disks pull back via f to disjoint disks $E_1, E_2 \subset E(\tau)$. Each E_i thus provides an isotopy of $K_i \cap E(\tau)$ to $\partial N(\tau)$, making $(E(\tau), K \cap E(\tau))$ a rational tangle. Meanwhile, $N(\tau)$

intersects K in two short arcs that are clearly boundary-parallel, so that is a rational tangle too. Therefore, K is a 2-bridge knot with splitting sphere $\partial N(\tau)$.

It is known (from [9], for example) that a splitting of a 2-bridge knot into rational tangles is unique up to isotopy. Thus an ambient isotopy of S^3 carries K to a 4-plat projection and $N(\tau)$ to a half-space containing just the two maxima (or minima). Thus τ is an upper or lower tunnel for K. $\square$

4. Cyclic Groups Acting on Handlebodies

A key ingredient in the proof of Theorem 1.1 is the equivariant loop theorem of Meeks and Yau.

Theorem 4.1. [5] *Let M be a compact, connected 3-manifold with connected boundary, and G be a finite group acting smoothly on M. Let H be the kernel of the homomorphism of fundamental groups induced by the inclusion $\partial M \hookrightarrow M$. Then there is a collection $\Delta = \{D_1, \ldots, D_n\}$ of disjoint, properly embedded, essential disks in M such that*

(1) *$\partial D_i \subset \partial M$ for all i,*
(2) *H is the normal subgroup of $\pi_1(M)$ generated by the ∂D_i, and*
(3) *Δ is invariant under the action of G.*

Notation. Let M and Δ be as above, and denote by S be the union of the disks in Δ. Let $M|\Delta = M \smallsetminus N(S)$, where the regular neighborhood is chosen to be invariant under the action of G.

To adapt this theorem for our purposes, we assume that M is a handlebody and analyze the pieces of $M|\Delta$.

Lemma 4.2. *Let V be a handlebody, and let G be a finite group acting smoothly on V. Let Δ be the collection of disks guaranteed by Theorem 4.1. Then each component of $V|\Delta$ is a ball.*

Proof. First, suppose that V is cut along just one disk D. It is well-known that the result is one or two handlebodies. Here is an outline of the proof, suggested by Saul Schleimer. Let V' be one component of the resulting manifold. Then $\pi_1(V')$ injects into $\pi_1(V)$, because if a loop $\gamma \subset V'$ bounds a disk $E \subset V$, one can do disk swaps with D to find a disk that γ bounds in V' also. Thus $\pi_1(V')$ is free (as a subgroup of the free group $\pi_1(V)$), and so V' is a handlebody. Applying this argument repeatedly, we see that every component of $V|\Delta$ is a handlebody.

Let X be one component of $V|\Delta$. ∂X consists of a subset of ∂V along with some number of distinguished disks $E_1, \ldots, E_k$ that come

from removing $N(S)$. We already know that X is a handlebody, and we prove that X is a ball by showing that it contains no essential disks.

Suppose that $D \subset X$ is a disk with $\gamma = \partial D \subset \partial X$. Clearly, D can be isotoped so that γ is disjoint from the distinguished disks E_j. Thus γ is a simple closed curve in ∂V that bounds a disk $D \subset V$. By Theorem 4.1, some loop freely homotopic to γ is generated by conjugates of the ∂D_i in $\pi_1(\partial V)$. Passing to homology, we see that the class $[\gamma]$ is a linear combination of the $[\partial D_i]$ in $H_1(\partial V)$.

Let us express ∂V as a union of open subsets A and B, where $B = \partial V \setminus \partial X$ and A is an open regular neighborhood of ∂X in ∂V. Thus $A \cap B$ is a disjoint union of open regular neighborhoods of the ∂E_j. γ lies in A but is homologous to some cycle c in B, because

$$[\gamma] = \sum_{i=1}^{n} a_i [\partial D_i] \in H_1(\partial V), \text{ and } \partial D_i \subset B \text{ for all } i.$$

Now, the Mayer-Vietoris sequence gives us

$$\ldots \to H_1(A \cap B) \xrightarrow{i_*} H_1(A) \oplus H_1(B) \xrightarrow{j_*} H_1(\partial V) \to \ldots$$

induced by the chain maps $i(x) = (x, x)$ and $j(x, y) = x - y$. Since $[\gamma] - [c] = 0 \in H_1(\partial V)$, $([\gamma], [c]) \in \ker(j_*)$. But since the sequence is exact, $([\gamma], [c]) \in \mathrm{Im}(i_*)$. Thus γ is homologous in $H_1(A)$ to a cycle that lies in $A \cap B$. But $H_1(A \cap B)$ is generated by the $[\partial E_j]$, so γ must be homologically trivial in ∂X.

Since X is a handlebody without any homologically essential disks, it must be a ball. □

The following application of Meeks and Yau's theorem to a form of the Smith Conjecture for handlebodies was suggested by Ian Agol and Saul Schleimer.

Theorem 4.3. *Let V be a handlebody, and let $g : V \to V$ be an orientation-preserving, periodic diffeomorphism. Then*

(1) *The fixed-point set of g is either empty or boundary-parallel.*
(2) *$W = V/(g)$ is also a handlebody, in which the image of the fixed-point set is again empty or boundary-parallel.*

Proof. Let Δ be the collection of disks in V given by Theorem 4.1. By Lemma 4.2, $V|\Delta$ is a disjoint union of balls. The fixed-point set a may be empty, or it may have one or more components. If a is non-empty, we will prove that it is boundary-parallel by first considering its intersection with the individual balls and then seeing how the pieces join up.

For each component B of $V|\Delta$, ∂B consists of a subset of ∂V, together with some number of distinguished disks $E_1, \ldots, E_k$ that come

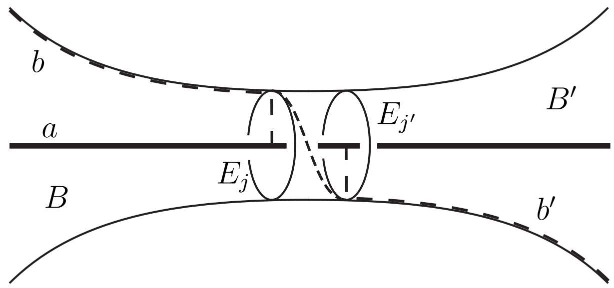

FIGURE 3. When the components of $V|\Delta$ are glued together along Δ, the fixed-point locus a remains isotopic to the boundary.

from removing $N(S)$. Since $V|\Delta$ is g-invariant, g maps B either onto itself or onto another ball. Thus if $B \cap a$ is non-empty, g must send B to itself.

Double B along its boundary to get S^3. If g maps B to itself, its action will extend to an orientation-preserving, periodic, smooth map $h : S^3 \to S^3$. By the solution to the Smith Conjecture [6], the fixed-point set of h is a single unknotted circle. Thus the double of $B \cap a$ is the unknot, making $B \cap a$ a single, boundary-parallel arc. We can choose the isotopic arc $b \subset \partial B$ so that it intersects the E_j in a minimal way: if an endpoint of $B \cap a$ lies in some E_j, then $b \cap E_j$ is a radius of that disk. Otherwise, b is disjoint from the distinguished disks.

To prove that a is boundary-parallel in all of V, it remains to show how the isotopies in the individual balls extend across the disks in Δ. If some $D_i \in \Delta$ is disjoint from a, it presents no problem. If D_i does intersect a, it is preserved by g, and the intersection must be a point or an arc. If D_i intersects a in an arc, then that component of a is already boundary-parallel through D_i. If D_i intersects a transversely, in a point, then the corresponding pair of disks $E_j \subset \partial B$ and $E_{j'} \subset \partial B'$ will each intersect it in a point too. In this case, the isotopies of $a \cap B$ and $a \cap B'$ to ∂V can be joined over $N(D_i)$, as in Figure 3. Thus any non-empty fixed-point locus a must be boundary-parallel.

To prove part (2) of the theorem, consider the quotient of $V|\Delta$ under the action of g. This quotient is still a disjoint union of balls, which are glued along disks on their boundaries to reconstruct W. Thus W can be viewed as a thickened graph, whose vertices are in the disjoint balls and whose edges correspond to gluings. As a thickened graph, W must be a handlebody, and the image of a under the quotient map is boundary-parallel by the same argument as above. $\square$

5. Proof of Theorem 1.1

Lemma 3.1 proves the "if" direction of the theorem. To prove the "only if" direction, suppose that a strong involution ψ of K fixes its unknotting tunnel τ pointwise. By the Smith Conjecture solution, the fixed-point locus of ψ is an unknotted circle. τ already lies on this axis; call the remaining arc of the axis a.

Notation. Let $\pi : S^3 \to S^3$ be the quotient map induced by the action of ψ. Label $\hat{K} = \pi(K)$, $\hat{\tau} = \pi(\tau)$, and $\hat{a} = \pi(a)$. Then π is a branched covering map of S^3 by S^3, branched along the unknot $\hat{\tau} \cup \hat{a}$.

Recall the genus-2 Heegaard splitting of S^3 by $V_1 = \overline{N(K \cup \tau)}$ and $V_2 = E(K \cup \tau)$. ψ acts as an involution on each V_i; let $W_i = \pi(V_i)$ be the quotients. Since W_1 is a closed regular neighborhood of the knot $\hat{K} \cup \hat{\tau}$, it is a solid torus. By Theorem 4.3, W_2 is a handlebody, and since $T = \partial W_1 = \partial W_2$ is a torus, W_2 is itself a solid torus. The result now follows in two steps.

Claim 5.1. *$\hat{K} \cup \hat{a}$ is a two-bridge knot with splitting sphere $\partial N(\hat{\tau})$.*

Proof. The involution ψ acts on each V_i separately, and a is the fixed-point set of ψ in V_2. By Theorem 4.3, it follows that $\hat{a}$ is boundary-parallel in W_2. Thus $W_2 \setminus N(\hat{a}) = E(\hat{K} \cup \hat{\tau} \cup \hat{a})$ is a genus-2 handlebody, and $\hat{\tau}$ is an unknotting tunnel for $\hat{K} \cup \hat{a}$. Furthermore, $\hat{\tau} \cup \hat{a}$ is the unknot by the solution to the Smith Conjecture, and $\hat{K} \cup \hat{\tau}$ is the unknot because $W_2 = E(\hat{K} \cup \hat{\tau})$ is a solid torus. Thus Corollary 3.3 tells us that $\partial N(\hat{\tau})$ splits $\hat{K} \cup \hat{a}$ into rational tangles. □

Claim 5.2. *K is a two-bridge knot, and τ is its upper or lower tunnel.*

Proof. Claim 5.1 tells us that $\partial N(\hat{\tau})$ is a splitting sphere for $\hat{K} \cup \hat{a}$. In particular, $\hat{K} \cap E(\hat{\tau})$ is isotopic to $\partial N(\hat{\tau})$ via a disk $\hat{D}$ that is disjoint from $\hat{a}$. Since $\hat{D}$ is disjoint from the branch locus of π, it lifts to two disjoint disks, D_1 and D_2, that realize isotopies of K_1 and K_2, respectively, to $\partial N(\tau)$. Therefore, $(E(\tau), K \cap E(\tau))$ is a rational tangle and K is a 2-bridge knot.

It follows that τ is an upper or lower tunnel for K, by the same argument as in Corollary 3.3. □

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